

# Abundant $p$ -singular elements in finite classical groups

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## Abstract

In 1995, Isaacs, Kantor and Spaltenstein proved that for a finite simple classical group  $G$  defined over a field with  $q$  elements, and for a prime divisor  $p$  of  $|G|$  distinct from the characteristic, the proportion of  $p$ -singular elements in  $G$  (elements with order divisible by  $p$ ) is at least a constant multiple of  $(1 - 1/p)/e$ , where  $e$  is the order of  $q$  modulo  $p$ . Motivated by algorithmic applications, we define a subfamily of  $p$ -singular elements, called  $p$ -abundant elements, which leave invariant certain “large” subspaces of the natural  $G$ -module. We find explicit upper and lower bounds for the proportion of  $p$ -abundant elements in  $G$ , and prove that it approaches a (positive) limiting value as the dimension of  $G$  tends to infinity. It turns out that the limiting proportion of  $p$ -abundant elements is at least a constant multiple of the Isaacs–Kantor–Spaltenstein lower bound for the proportion of *all*  $p$ -singular elements.

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## 1 Introduction

Given a prime  $p$  dividing the order of a finite group  $G$ , what proportion of elements in  $G$  are  $p$ -singular? That is, what proportion of elements in  $G$  have order divisible by  $p$ ? Isaacs et al. [4] considered this problem for permutation groups of degree  $n$  and proved that the proportion is at least  $1/n$ . At the heart of their proof is consideration of the case where  $G$  is a finite simple group of Lie type, and more particularly a finite simple  $d$ -dimensional classical group. In this case they obtained for the proportion of  $p$ -singular elements in  $G$  a lower bound of the form  $(1 - 1/p)c/d$  for some constant  $c$ , independently of the type

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of  $G$  and of the size  $q$  of the field over which  $G$  is defined. A closer inspection of their proof reveals that their lower bound can in fact be written as a constant times  $(1 - 1/p)/e$ , where  $e$  is the order of  $q$  modulo  $p$  (the least positive integer for which  $p$  divides  $q^e - 1$ ); we thank Klaus Lux for pointing this out to us.

Estimates for proportions of  $p$ -singular elements are important in complexity analyses of numerous algorithms in computational group theory. In particular, the first and third authors' algorithm [11] for recognising finite classical groups in their natural representations relies on finding by repeated independent random selection from  $G$  elements with orders divisible by primes  $p$  for which  $e$  is greater than half the dimension  $d$ , and there exists an efficient practical algorithm for testing whether an element has this property [10]. However, these particular  $p$ -singular elements are relatively scarce: they arise with frequency proportional to  $1/e = O(1/d)$ , whereas the work of Isaacs et al. [4] suggests that, in general,  $p$ -singular elements are more frequent when  $e$  is smaller. Moreover, the restriction  $e > d/2$  only allows us to identify elements with orders divisible by *certain* primes, namely those primes  $p$  for which the order of  $q$  modulo  $p$  is greater than  $d/2$ .

These shortcomings motivated us to seek, for *all* values of  $e$ , a class of  $p$ -singular elements that arise with frequency proportional to  $1/e$  and can be efficiently recognised algorithmically, with the hope being that such elements might lead to improved recognition algorithms for finite classical groups. Experimental evidence gathered by the first author suggested that a particular type of  $p$ -singular element, which we term *p-abundant* (see Definition 1.1), arises approximately with frequency proportional to  $1/e$ . The theoretical analysis presented in this paper proves that this is indeed the case, for the groups in Table 1. Moreover, the  $p$ -abundant elements are indeed readily identifiable computationally; for example, from their characteristic polynomials. Algorithms for this are presented in a companion paper [9].

Our estimates for the proportion of  $p$ -abundant elements are very precise: we determine in Theorem 1.2 both upper and lower bounds and, in particular, the exact asymptotic value for this proportion. Precision of this kind seems to be rare in the literature: estimates for the proportions of various kinds of elements in finite groups tend to focus on lower bounds, and good upper bounds are rarely given. Our endeavour to obtain such precise bounds drove the development of the methodology presented in the first and third authors' paper [12], which underlies the proofs in this paper. This theory was in turn inspired by and developed from the methods used by Isaacs et al. [4] and an earlier application by the first and third authors in collaboration with Lübeck [7]. We note that this method was used earlier by Lehrer [5, 6] to study the representations of finite Lie type groups.

Our method requires us to sum over the lengths of certain conjugacy classes of the corresponding Weyl group, weighted by proportions of  $p$ -abundant elements in matching maximal tori. Whereas previous applications [7, 8] approximated the corresponding expressions by replacing all weighting factors with a common lower bound, here we have to be much more careful with our estimates. Our results highlight the power of the method of [12] to obtain exact asymp-

$G$	$\delta$	$d$
$\mathrm{GL}_n(q)$	1	$n$
$\mathrm{GU}_n(q)$	2	$n$
$\mathrm{Sp}_{2n}(q)$	1	$2n$
$\mathrm{SO}_{2n+1}(q)$	1	$2n+1$
$\mathrm{SO}_{2n}^{\pm}(q)$	1	$2n$

Table 1: The finite classical groups considered in this paper.

otic values for element proportions, and rely on some delicate technical lemmas. They constitute the first application of this theory achieving such precision.

We now formally define  $p$ -abundant elements in finite classical groups and state our results concerning proportions of these elements.

**Definition 1.1** Let  $q$  be a prime power,  $n$  a positive integer, and  $G, \delta, d = d(n)$  as in one of the lines of Table 1. Let  $V = V(d, q^\delta)$  denote the natural  $G$ -module. Let  $p$  be a prime dividing  $|G|$  and coprime to  $q$ , and  $m$  an integer with  $d/2 < m \leq d$ . An element  $g \in G$  is said to be  $(p, m)$ -abundant if, in its action on  $V$ ,  $g$  has an eigenvalue  $\zeta$  in some extension field of  $\mathbb{F}_{q^\delta}$  such that  $\zeta$  has multiplicative order divisible by  $p$  and either

- (i)  $\zeta$  has  $m$  Galois conjugates over  $\mathbb{F}_{q^\delta}$ , or
- (ii)  $G \neq \mathrm{GL}_n(q)$ ,  $m$  is even,  $\zeta$  and  $\zeta^{-1}$  are not Galois conjugate, and  $\zeta$  and  $\zeta^{-1}$  have together  $m$  Galois conjugates over  $\mathbb{F}_{q^\delta}$ .

The element  $g$  is called  $(p, m)$ -abundant *irreducible* in case (i), and  $(p, m)$ -abundant *quasi-irreducible* in case (ii). In either case, a  $p$ -abundant element is one which is  $(p, m)$ -abundant for some  $m$  with  $d/2 < m \leq d$ .

The terms “irreducible” and “quasi-irreducible” are chosen to reflect certain properties of the actions of  $p$ -abundant elements on the natural  $G$ -module. The  $(p, m)$ -abundant irreducible elements leave invariant a unique irreducible subspace of dimension  $m$ . In particular, we note that the  $p$ -abundant irreducible elements contain the family of so-called *primitive prime divisor* elements which underly the first and third authors’ classical recognition algorithm [11]. The  $(p, m)$ -abundant quasi-irreducible elements have a similar property, preserving a specific decomposition of a unique invariant  $m$ -dimensional subspace into two closely related irreducible subspaces of dimension  $m/2$ . The proofs of these facts are omitted here for brevity, but may be found in our related paper [9] concerning algorithms for identifying  $p$ -abundant elements computationally; see also the papers by Huppert [2, 3].

**Theorem 1.2** Let  $q$  be a prime power,  $n$  an integer with  $n \geq 9$ , and  $G$  as in one of the lines of Table 2. Suppose that  $p$  is an odd prime dividing  $|G|$  and

line	$G$	$T$	$e$ parity	$e$ range	$c$	$\kappa$	$\alpha$	$\beta$
1	$GL_n(q)$	$I$	all	$e > n/p$	1	0	1	0
2				$e \leq n/p$	1	1	1	1
3	$GU_n(q)$	$I$	$2 \pmod{4}$	$e > 2n/p$	1	0	2	0
4				$e \leq 2n/p$	1	1	2	1
5		$QI$	even	$e > n/p$	1	0	1	0
6				$e \leq n/p$	1	1	1	1
7			odd	$e > n/p$	1/2	0	1	0
8				$e \leq n/p$	1/2	1	1	1/2
9	$Sp_{2n}(q)$	$I$	even	$e > 2n/p$	1/2	0	3/2	0
10	$SO_{2n+1}(q)$			$e \leq 2n/p$	1/2	1	3/2	1/2
11	$SO_{2n}^\pm(q)$	$QI$	all	$e > n/p$	1/2	0	1	0
12				$e \leq n/p$	1/2	1	1	1/2

Table 2: Cases for Theorem 1.2. (Line numbers are listed for later reference.)

coprime to  $q$ . Let  $e$  denote the smallest positive integer such that  $p$  divides  $q^e - 1$ , and  $t$  the largest integer such that  $p^t$  divides  $q^e - 1$ . Let  $Q(p; I; G)$  denote the set of all  $p$ -abundant irreducible elements in  $G$ , and  $Q(p; QI; G)$  the set of all  $p$ -abundant quasi-irreducible elements. Then for  $T \in \{I, QI\}$  and constants  $c, \kappa, \alpha, \beta$  depending on  $G$  and  $e$  as in Table 2, we have

$$-\left(\alpha + \frac{\beta \ln(2)}{p^t}\right) \frac{1}{n} - \frac{3 \ln(2)}{eq^{n/4}} < \frac{|Q(p; T; G)|}{|G|} - \left(1 - \frac{1}{p^{t-1}(p + \kappa)}\right) \cdot \frac{c \ln(2)}{e} \leq \frac{\alpha}{n}. \quad (1.1)$$

The proof of Theorem 1.2 is given in Section 3, using preliminary theoretical results summarised in Section 2 and technical lemmas collected in Section 4. We note that we also take the opportunity to mention a small improvement to the results of our aforementioned paper [8] in Remark 2.5 (that paper is unrelated to the present one, but also relies on the theory outlined in Section 2). Here we just make a few remarks about Theorem 1.2.

**Remark 1.3** (a) For combinations of  $T$  and  $e$  not appearing in Table 2, the set  $Q(p; T; G)$  is empty. From Definition 1.1, the quasi-irreducible case does not arise in  $GL_n(q)$ . For  $GU_n(q)$ ,  $p$ -abundant irreducible elements arise only when  $e \equiv 2 \pmod{4}$ , and in the symplectic and orthogonal groups, the irreducible case arises only when  $e$  is even. The details are given in Section 3.

(b) A perhaps surprising consequence of Theorem 1.2 is that the proportion of  $p$ -abundant elements is at least a constant multiple of the lower bound obtained by Isaacs et al. [4] for the proportion of *all*  $p$ -singular elements in  $G$ . Specifically, upon observing that  $1/(p^{t-1}(p + \kappa)) \leq 1/p$  in (1.1), we obtain

$$\frac{|Q(p; T; G)|}{|G|} > \left(1 - \frac{1}{p}\right) \frac{c'}{e}$$

for some constant  $c'$ , in all cases where  $Q(p; \mathbf{T}; G)$  is nonempty.

(c) We do not consider the prime  $p = 2$ , for which the results would be a little different (Lemma 4.1 would need modification, amongst other things). We do, however, believe that a similar result holds in this case.

(d) The assumption  $n \geq 9$  is made for technical reasons, as certain inequalities used in deriving (1.1) are invalid for very small values of  $n$  (see Lemmas 4.5 and 4.6(ii)). However, in proving Theorem 1.2 we obtain general closed-form expressions for proportions of  $p$ -abundant elements, given in equations (3.2), (3.9), (3.11) and (3.12). These expressions depend on certain auxiliary quantities which we estimate (using Lemma 4.3) in order to obtain the bounds given in the theorem. But in principle, they can be used to calculate proportions of  $p$ -abundant elements exactly, at least in certain simple cases. In addition to the small  $n$  cases not covered by (1.1), we have in mind situations where  $e$  is reasonably large, say at least a constant fraction of  $n$ . An example, where  $G = \mathrm{GL}_n(q)$  with  $e \geq n/2$ , is discussed in Remark 3.1 as illustration.

## 2 Strategy

Throughout the paper we use the following hypothesis.

**Hypothesis 2.1** Let the group  $G$ , its dimension  $d$  and the value of  $\delta$  be as in one of the lines of Table 1, and let  $V = V(d, q^\delta)$  be the natural  $G$ -module. Assume that we have obtained  $G$  as the fixed point set  $\hat{G}^F$  of a connected reductive algebraic group  $\hat{G}$  defined over the algebraic closure  $\bar{\mathbb{F}}_q$  of  $\mathbb{F}_q$ , with  $F$  a Frobenius morphism of  $\hat{G}$ . Moreover, assume that  $F$  and a maximal torus  $T_0$  in  $\hat{G}$  have been chosen in the same way as outlined in [7, Section 3], so that  $W = N_{\hat{G}}(T_0)/T_0$  is the corresponding Weyl group.

### 2.1 Estimating proportions via quokka sets

In order to derive upper and lower bounds for proportions of  $p$ -abundant elements in the groups  $G$  listed in Table 1, we apply the theory of *quokka sets* of finite groups of Lie type [7, 12]. These are subsets whose proportion in  $G$  can be derived by determining certain proportions in maximal tori in  $G$  and certain proportions in the corresponding Weyl group.

Recall [1, p. 11] that each element  $g \in G$  has a unique Jordan decomposition  $g = su$ , where  $s \in G$  is semisimple,  $u \in G$  is unipotent and  $su = us$ , with  $s$  called the *semisimple part* of  $g$  and  $u$  the *unipotent part*. Note that the order  $o(s)$  of  $s$  is coprime to the characteristic, and  $o(u)$  is a power of the characteristic.

The concept of a quokka set is introduced for finite groups of Lie type in [12, Definition 1.1]. A nonempty subset  $Q$  of one of the groups  $G$  in Table 1 is a *quokka set* if the following two conditions hold:

- (i) if  $g \in G$  has Jordan decomposition  $g = su$  with semisimple part  $s$  and unipotent part  $u$ , then  $g \in Q$  if and only if  $s \in Q$ ;

(ii)  $Q$  is a union of  $G$ -conjugacy classes.

We assume that Hypothesis 2.1 holds, and summarise the required results. A subgroup  $H$  of the connected reductive algebraic group  $\hat{G}$  is said to be  $F$ -stable if  $F(H) = H$ , and for each subgroup  $H$  of  $\hat{G}$  we write  $H^F = H \cap G^F$ . We define an equivalence relation on  $W$  as follows: elements  $w, w' \in W$  are  $F$ -conjugate if there exists  $x \in W$  such that  $w' = x^{-1}wF(x)$ . The equivalence classes of this relation on  $W$  are called  $F$ -conjugacy classes [1, p. 84]. The  $G$ -conjugacy classes of  $F$ -stable maximal tori are in one-to-one correspondence with the  $F$ -conjugacy classes of the Weyl group  $W$  of  $\hat{G}$ . The explicit correspondence is given in [1, Proposition 3.3.3].

Let  $\mathcal{C}$  be the set of  $F$ -conjugacy classes in  $W$  and, for each  $C \in \mathcal{C}$ , let  $T_C$  be a representative element of the family of  $F$ -stable maximal tori corresponding to  $C$ . The following theorem is a direct consequence of [12, Theorem 1.3].

**Theorem 2.2** *If Hypothesis 2.1 holds and  $Q \subseteq G$  is a quokka set, then*

$$\frac{|Q|}{|G|} = \sum_{C \in \mathcal{C}} \frac{|C|}{|W|} \frac{|T_C^F \cap Q|}{|T_C^F|}.$$

We refer to an  $F$ -stable maximal torus containing an element of  $Q$  as a *quokka torus*, and call the corresponding  $F$ -conjugacy classes of  $W$  *quokka classes*.

In order to apply Theorem 2.2, we check that the  $p$ -abundant elements in  $G$  form quokka sets. We introduce the following notation, similar to that used in Theorem 1.2 (for suitable  $p, m$ ):  $Q(p, m; \mathbf{I}; G)$  denotes the set of all  $(p, m)$ -abundant irreducible elements in  $G$ , and  $Q(p, m; \mathbf{QI}; G)$  the set of all  $(p, m)$ -abundant quasi-irreducible elements. For brevity we combine this notation as  $Q(p, m; \mathbf{T}; G)$ , where  $\mathbf{T}$  is one of the symbols  $\mathbf{I}$ ,  $\mathbf{QI}$  (as in Theorem 1.2). We have the following lemma.

**Lemma 2.3** *Suppose that Hypothesis 2.1 holds. Let  $p$  be a prime that is coprime to  $q$ ,  $\mathbf{T} \in \{\mathbf{I}, \mathbf{QI}\}$ , and  $m$  an integer with  $d/2 < m \leq d$  such that  $Q(p, m; \mathbf{T}; G)$  is nonempty. Then  $Q(p, m; \mathbf{T}; G)$  is a quokka set.*

**Proof.** It is clear from Definition 1.1 that each  $Q(p, m; \mathbf{T}; G)$  is a union of  $G$ -conjugacy classes. The condition that  $g$  lies in  $Q(p, m; \mathbf{T}; G)$  if and only its semisimple part  $s$  does follows from the well-known fact that  $g$  and  $s$  share the same characteristic polynomial.  $\square$

Since the requirement that  $m > d/2$  implies that an element can be  $(p, m)$ -abundant for at most one value of  $m$ , the sets  $Q(p; \mathbf{T}; G)$  in Theorem 1.2 are then the disjoint unions of the respective  $Q(p, m; \mathbf{T}; G)$  over all  $m$  with  $d/2 < m \leq d$ . Hence we have the following immediate corollary.

**Corollary 2.4** *The nonempty  $Q(p; \mathbf{T}; G)$  are quokka sets and satisfy*

$$|Q(p; \mathbf{T}; G)| = \sum_{d/2 < m \leq d} |Q(p, m; \mathbf{T}; G)|.$$

## 2.2 Maximal tori of the groups in Table 1

Suppose that Hypothesis 2.1 holds. In order to apply Theorem 2.2 to estimate the proportion of elements in  $G$  that lie in  $Q(p; T; G)$  for some odd prime  $p$  dividing  $|G|$  and not dividing  $q$ , we have to describe the  $F$ -conjugacy classes of the Weyl group  $W$  and their corresponding maximal tori. We summarise the description given in [7] and [12], where more details can be found.

Consider first the cases where  $G = \mathrm{GL}_n(q)$  or  $G = \mathrm{GU}_n(q)$ . Write  $\delta = 1$ ,  $\epsilon = 1$  in the first case and  $\delta = 2$ ,  $\epsilon = -1$  in the second. Note that the Weyl group  $W$  is isomorphic to  $S_n$  via an isomorphism  $W \rightarrow S_n$  that we denote (for reference in Section 3.1) by  $\sigma$ . The  $F$ -conjugacy classes of  $W$  are the conjugacy classes of  $W$ . So the  $F$ -conjugacy classes are parameterised by the partitions of  $n$  describing the cycle types of permutations in  $S_n$ . If  $w \in W$  corresponds to the partition  $\mu = (m_1, \dots, m_k)$  of  $n$  then each maximal torus  $T^F$  of  $G$  corresponding to the conjugacy class of  $w$  is isomorphic to

$$\mathbb{Z}_{q^{m_1 - \epsilon^{m_1}}} \times \cdots \times \mathbb{Z}_{q^{m_k - \epsilon^{m_k}}}.$$

A cyclic direct factor  $\mathbb{Z}_{q^m \pm 1}$  of  $T^F$  corresponds to elements that have  $m$  eigenvalues in  $\bar{\mathbb{F}}_{q^\delta}$  that lie in  $\mathbb{F}_{q^{\delta m}}$  and are permuted by the map  $a \mapsto a^{\epsilon q}$ . In particular, if  $m > d/2$  and  $T^F$  contains  $(p, m)$ -abundant elements, then for such elements these  $m$  eigenvalues are precisely the Galois conjugates of  $\zeta$  and  $\zeta^{-1}$  as described in Definition 1.1. (Note here that for  $G = \mathrm{GU}_n(q)$ , and similarly for the symplectic and orthogonal groups discussed below, if  $\zeta$  is an eigenvalue of  $g \in G$  then  $\zeta^{-1}$  is also an eigenvalue, because  $g$  is conjugate to its inverse transpose via the matrix of the form preserved by  $G$ .)

Now consider  $G = \mathrm{Sp}_{2n}(q)$  or  $G = \mathrm{SO}_{2n+1}(q)$ . Here the Weyl group  $W$  is isomorphic to  $S_2 \wr S_n$ , acting imprimitively on the set  $\Omega = \{\pm 1, \dots, \pm n\}$  of size  $2n$ , and consists of the so-called *signed permutations*; that is to say, for  $i, j \in \Omega$  and  $g \in W$ ,  $i^g = j$  if and only if  $(-i)^g = -j$ . We define a projection  $\sigma : W \rightarrow S_n$  by mapping a signed permutation to the permutation it induces on  $\{1, \dots, n\}$ . For  $g \in W$ , a cycle of the image  $\sigma(g)$  with length  $\lambda$  is *positive* if it is the image under  $\sigma$  of two  $g$ -cycles in  $\Omega$  of length  $\lambda$ , and *negative* if it is the image under  $\sigma$  of one  $g$ -cycle in  $\Omega$  of length  $2\lambda$ . A conjugacy class of  $W$  is determined by its cycle type in  $S_n$  and the numbers of positive cycles of each length. Suppose that  $(\mu^+, \mu^-)$  is a partition of  $n$  that determines a conjugacy class whose positive cycle lengths make up the parts of  $\mu^+ = (m_1^+, \dots, m_j^+)$  and negative cycle lengths make up the parts of  $\mu^- = (m_1^-, \dots, m_k^-)$ . Then each corresponding maximal torus  $T^F$  of  $G$  is isomorphic to

$$\left( \prod_{i=1}^j \mathbb{Z}_{q^{m_i^+} - 1} \right) \times \left( \prod_{i=1}^k \mathbb{Z}_{q^{m_i^-} + 1} \right). \quad (2.1)$$

Here a cyclic factor  $\mathbb{Z}_{q^\lambda \pm 1}$  corresponds to elements that have  $m = 2\lambda$  eigenvalues in  $\bar{\mathbb{F}}_q$  that lie in  $\mathbb{F}_{q^m}$  and are permuted by the map  $a \mapsto a^q$ . If  $m > d/2$  and  $T^F$  contains  $(p, m)$ -abundant elements, then for such elements these  $m$  eigenvalues are precisely the Galois conjugates of  $\zeta$  and  $\zeta^{-1}$  as in Definition 1.1.

Finally, consider  $G = \mathrm{SO}_{2n}^{\pm}(q)$ . We can view this group as a subgroup of  $\mathrm{SO}_{2n+1}(q)$ . The Weyl group  $W$  has index 2 in the Weyl group of  $\mathrm{SO}_{2n+1}(q)$ , which we denote by  $W_B$  below; namely,  $W$  is the intersection of  $W_B$  with the alternating group on  $\Omega$ . An element  $w \in W_B$  lies in  $W$  if and only if it has an even number of negative cycles. Moreover, we choose an element  $w_n \in W_B$  such that  $W_B = W \dot{\cup} Ww_n$ , as described in [7, Section 3.4]. The  $F$ -conjugacy classes of  $W$  correspond to partitions  $(\mu^+, \mu^-)$  of  $n$  such that  $\mu^-$  has an even number of parts in the case  $G = \mathrm{SO}_{2n}^+(q)$  or an odd number of parts (and hence  $|\mu^-| > 0$ ) in the case  $G = \mathrm{SO}_{2n}^-(q)$ . The corresponding maximal tori are isomorphic to the groups in the product displayed in (2.1), and similar comments about  $p$ -abundant elements in these tori apply.

**Remark 2.5** We take this opportunity to mention a small improvement to our paper [8], which is also based on the *quokka theory* outlined above. The lower bounds obtained in that paper for the proportions of so-called *pre-involutions* in finite classical groups can in fact be multiplied by 2 in the cases  $G = \mathrm{SO}_{2n}^{\pm}(q)$ . Specifically, the ‘1/4’ in the last line of [8, Table 1] may be replaced by ‘1/2’, and the right-hand sides of the inequalities in [8, Theorem 1.5(iii) and Corollary 1.6(ii)] may be multiplied by 2. In the arguments in [8, Section 4.6] for the cases  $G = \mathrm{SO}_{2n}^{\pm}(q)$ , the result [8, Lemma 2.3] should have been applied in conjunction with the fact (mentioned on [8, p. 1025]) that the Weyl group of type  $D_n$  has index 2 in the Weyl group of type  $B_n$ , as in the proof of [12, Lemma 4.2]; this would have yielded the additional factor of 2 in our lower bounds.

### 3 Proof of Theorem 1.2

We now prove Theorem 1.2 using the strategy described in Section 2. To aid the exposition, we refer in several places to technical results whose proofs are given in Section 4. The following notation is used frequently: for a positive integer  $k$  and a prime  $r$ ,  $(k)_r$  denotes the highest power of  $r$  that divides  $k$ .

#### 3.1 $G = \mathrm{GL}_n(q)$

First suppose that  $G = \mathrm{GL}_n(q)$ . According to Definition 1.1, only the irreducible case ( $\mathbf{T} = \mathbf{I}$ ) arises. Here  $C$  is a quokka class for  $Q(p; \mathbf{I}; G)$  if and only if  $T_C^F$  contains a direct factor  $A_m \cong \mathbb{Z}_{q^m-1}$  and  $p$  divides  $q^m - 1$  for some  $m$  with  $n/2 < m \leq n$ . By Lemma 4.1(i),  $p$  divides  $q^m - 1$  if and only if  $e$  divides  $m$ , where  $e$  is the order of  $q$  modulo  $p$ . Thus, for an element  $g$  of  $C$ , the image  $\sigma(g) \in S_n$  has at least one cycle of length  $m = be$ , for some  $b$ , with  $n/2 < m \leq n$ , namely  $n/(2e) < b < m/e$ . Note that any permutation in  $S_n$  can have at most one such cycle and the proportion of elements in  $S_n$  with a cycle of length  $m$  is

$$\binom{n}{m} (m-1)! (n-m)! = \frac{1}{m}. \quad (3.1)$$

For each  $F$ -conjugacy class  $C$  with a cycle of length  $m$ , the maximal torus  $T_C^F$  of  $G$  corresponding to  $C$  can be expressed as  $T_C^F = A_m \times B$ , where  $B$  is a product



of cyclic groups of the form  $\mathbb{Z}_{q^{b_k}-1}$  with all  $b_k \leq d/2$  and hence contains no  $p$ -abundant elements.

Now, an element  $g$  of this maximal torus  $T_C^F$  is  $(p, m)$ -abundant (irreducible) if and only if, in its action on  $V$ ,  $g$  has an eigenvalue in some extension field of  $\mathbb{F}_q$  with multiplicative order divisible by  $p$  and  $m$  Galois conjugates over  $\mathbb{F}_q$ . So, by the discussion in Section 2.2,  $g$  is  $(p, m)$ -abundant irreducible if and only if its  $A_m$ -component has an eigenvalue in  $\mathbb{F}_{q^m}$  with  $m$  Galois conjugates over  $\mathbb{F}_q$  and order divisible by  $p$ . Hence, denoting by  $\theta(b)$  the proportion of elements in  $\mathbb{F}_{q^{be}}^*$  that have  $be$  Galois conjugates over  $\mathbb{F}_q$  and order divisible by  $p$ , we have the following explicit expression for the proportion of  $p$ -abundant irreducible elements in  $G = \mathrm{GL}_n(q)$ :

$$\frac{|Q(p; \mathbf{I}; G)|}{|G|} = \sum_{n/(2e) < b \leq n/e} \frac{\theta(b)}{be}. \quad (3.2)$$

An upper bound for  $\theta(b)$  is obtained by excluding the elements in  $\mathbb{F}_{q^{be}}^*$  with order not divisible by  $p$ . These comprise the unique subgroup of  $\mathbb{F}_{q^{be}}^*$  of index  $|\mathbb{F}_{q^{be}}^*|_p = (q^{be} - 1)_p$ . By Lemma 4.1(iii),  $(q^{be} - 1)_p = p^{t+j}$ , where  $p^j = (b)_p$  and  $p^t = (q^e - 1)$ , and hence  $\theta(b) \leq 1 - 1/p^{t+j}$ . A lower bound for  $\theta(b)$  is obtained by considering the proportion of elements in  $\mathbb{F}_{q^{be}}^*$  with  $be$  Galois conjugates, and then subtracting the proportion of elements in  $\mathbb{F}_{q^{be}}^*$  with order not divisible by  $p$ . A lower bound for the former proportion is given by Lemma 4.3(i) with  $\ell = be$ , yielding  $\theta(b) > 1 - 1/p^{t+j} - 3/q^{be/2}$ . Therefore, and since  $be = m > d/2 = n/2$ , we have

$$1 - \frac{1}{p^{t+j}} - \frac{3}{q^{n/4}} < \theta(b) \leq 1 - \frac{1}{p^{t+j}}. \quad (3.3)$$

We now estimate the sum in (3.2) to derive the bounds for  $|Q(p; \mathbf{I}; G)|/|G|$  claimed in Theorem 1.2. First suppose that  $e > n/p$ . Then all possible values of  $b$  in (3.2) satisfy  $b \leq n/e < p$  and hence have  $j = 0$  in the inequalities for  $\theta(b)$  in (3.3). That is, we have  $1 - 1/p^t - 3/q^{n/4} < \theta(b) \leq 1 - 1/p^t$ , independently of  $b$ . Then, using the notation and bounds of Lemma 4.4(i), we obtain

$$\begin{aligned} \frac{|Q(p; \mathbf{I}; G)|}{|G|} &\leq \left(1 - \frac{1}{p^t}\right) \frac{P(n/e, 1)}{e} \\ &\leq \left(1 - \frac{1}{p^t}\right) \left(\frac{\ln(2)}{e} + \frac{1}{n}\right) < \left(1 - \frac{1}{p^t}\right) \frac{\ln(2)}{e} + \frac{1}{n} \end{aligned} \quad (3.4)$$

and, similarly,

$$\frac{|Q(p; \mathbf{I}; G)|}{|G|} > \left(1 - \frac{1}{p^t} - \frac{3}{q^{n/4}}\right) \frac{P(n/e, 1)}{e} > \left(1 - \frac{1}{p^t}\right) \frac{\ln(2)}{e} - \frac{1}{n} - \frac{3 \ln(2)}{eq^{n/4}}. \quad (3.5)$$

Note that in (3.5) we require that  $1 - 1/p^t - 3/q^{n/4} > 0$ , which holds under the assumption  $n \geq 9$  made in Theorem 1.2 according to Lemma 4.5.

Now consider the case where  $e \leq n/p$ . Denote by  $i$  the positive integer satisfying  $p^i \leq n/e < p^{i+1}$ . Then each  $b$  in (3.2) satisfies  $(b)_p = p^j$  for some

$j \in \{0, \dots, i\}$ , and hence the bounds for  $\theta(b)$  given in (3.3) depend on the variable  $j = j(b)$  (unlike when  $e > n/p$ ). We take this dependence on  $j$  into account in order to obtain the precise leading term of  $|Q(p; \mathbf{I}; G)|/|G|$ . Write (3.2) as

$$\frac{|Q(p; \mathbf{I}; G)|}{|G|} = \sum_{j=0}^i \left( \sum_{\substack{n/(2e) < b \leq n/e \\ (b)_p = p^j}} \frac{\theta(b)}{be} \right). \quad (3.6)$$

Applying (3.3) and using the notation of Lemma 4.4(i) yields

$$\begin{aligned} \frac{|Q(p; \mathbf{I}; G)|}{|G|} &\leq \frac{1}{e} \sum_{j=0}^i \left( \left(1 - \frac{1}{p^{t+j}}\right) \sum_{\substack{n/(2e) < b \leq n/e \\ (b)_p = p^j}} \frac{1}{b} \right) \\ &\leq \frac{1}{e} \sum_{j=0}^{i-1} \left(1 - \frac{1}{p^{t+j}}\right) (P(n/e, p^j) - P(n/e, p^{j+1})) \\ &\quad + \frac{1}{e} \left(1 - \frac{1}{p^{t+i}}\right) P(n/e, p^i) \end{aligned} \quad (3.7)$$

and, similarly,

$$\begin{aligned} \frac{|Q(p; \mathbf{I}; G)|}{|G|} &> \frac{1}{e} \sum_{j=0}^{i-1} \left(1 - \frac{1}{p^{t+j}} - \frac{3}{q^{n/4}}\right) (P(n/e, p^j) - P(n/e, p^{j+1})) \\ &\quad + \frac{1}{e} \left(1 - \frac{1}{p^{t+i}} - \frac{3}{q^{n/4}}\right) P(n/e, p^i). \end{aligned} \quad (3.8)$$

The bounds asserted in Theorem 1.2 now follow upon application of Lemma 4.4(i), and of Lemma 4.6 with  $f_j = P(n/e, p^j)$  and hence  $\ell = n/e$ ,  $k_1 = \ln(2)$ ,  $k_2 = 1$  (and  $p, q, t, i$  as already defined). (Note that the assumption  $n \geq 9$  made in the theorem is used when applying Lemma 4.6(ii).)

**Remark 3.1** As mentioned in Remark 1.3(d), the closed-form expression (3.2) for the proportion of  $p$ -abundant irreducible elements in  $G = \mathrm{GL}_n(q)$  can, in principle, be used to compute this proportion exactly, at least in some simple cases. We have in mind situations where  $e$  is reasonably large. As illustration, consider the case where  $n/2 < e \leq n$ . The sum in (3.2) then ranges over  $b$  with  $1/2 < b < 2$ , so that  $b = 1$  is the only possible value, and one can check that  $\theta(1) = 1 - 1/p^t$  (by taking  $\ell = 1$  in Lemma 4.3(i) and noting that the proportion considered there is then equal to 1). For simplicity we do not use these facts in obtaining the estimates given in Theorem 1.2, but here they show that for  $n/2 < e \leq n$ , the proportion of  $p$ -abundant irreducible elements in  $\mathrm{GL}_n(q)$  is exactly  $(1 - 1/p^t)/e$ . Note that this particular case also follows from [11, Lemma 5.6]. Similar comments apply to equations (3.9), (3.11) and (3.12) below.

### 3.2 $G = \mathbf{GU}_n(q)$

Now take  $G = \mathbf{GU}_n(q)$ . Here  $C$  is a quokka class for  $Q(p; \mathbf{T}; G)$  if and only if  $T_C^F$  has a direct factor  $A_m \cong \mathbb{Z}_{q^m - (-1)^m}$  and  $p$  divides  $q^m - (-1)^m$  for some  $m$  with  $n/2 < m \leq n$ . So, depending on the parity of  $m$ , the image  $\sigma(g) \in S_n$  of an element  $g$  of  $C$  must have a (unique) cycle of length  $m$  as described below. In each case we obtain an expression analogous to (3.2).

If  $m$  is odd then we need  $p$  to divide  $q^m + 1$ . By Lemma 4.1(ii), this occurs if and only if  $e$  divides  $2m$  and  $e$  does not divide  $m$ . So  $e \equiv 2 \pmod{4}$  and  $m = be/2$  for some odd  $b$  with  $n/e < b \leq 2n/e$ . The  $(p, m)$ -abundant elements in  $T_C^F$  are of irreducible type, since condition (ii) of Definition 1.1 does not arise for  $m$  odd. An element of  $T_C^F$  is  $(p, m)$ -abundant irreducible if and only if its  $A_m$ -component has an eigenvalue in  $\mathbb{F}_{q^{2m}}^*$  with  $m$  Galois conjugates over  $\mathbb{F}_{q^2}$  and multiplicative order divisible by  $p$ . Hence, recalling from (3.1) the proportion of elements in  $S_n$  with an  $m$ -cycle, and denoting by  $\theta_1(b)$  the proportion of elements in  $\mathbb{Z}_{q^{be/2+1}} < \mathbb{F}_{q^{be}}^*$  with  $be/2$  Galois conjugates over  $\mathbb{F}_{q^2}$  and order divisible by  $p$ , we obtain the first case of (3.9) below.

If  $m$  is even then we need  $p$  to divide  $q^m - 1$ , which occurs if and only if  $e$  divides  $m$  (see Lemma 4.1(i)). Thus  $m = be$  for some  $b$  with  $n/(2e) < b \leq n/e$ , where  $b$  must be even if  $e$  is odd. The  $(p, m)$ -abundant elements in  $T_C^F$  are of quasi-irreducible type, and an element of  $T_C^F$  is  $(p, m)$ -abundant quasi-irreducible if and only if its  $A_m$ -component has an eigenvalue  $\zeta$  in  $\mathbb{F}_{q^m}^*$  with multiplicative order divisible by  $p$  such that  $\zeta$  and  $\zeta^{-1}$  are not Galois conjugate and have together  $m$  Galois conjugates over  $\mathbb{F}_{q^2}$ . Therefore, denoting by  $\theta_2(b)$  the proportion of elements in  $\mathbb{F}_{q^{be}}^*$  that have multiplicative order divisible by  $p$ , are not Galois conjugate to their inverses and have  $be/2$  Galois conjugates over  $\mathbb{F}_{q^2}$ , we obtain the second and third cases below:

$$\frac{|Q(p; \mathbf{T}; G)|}{|G|} = \begin{cases} \sum_{\substack{n/e < b \leq 2n/e \\ b \text{ odd}}} \frac{2\theta_1(b)}{be} & \text{if } e \equiv 2 \pmod{4} \text{ and } \mathbf{T} = \mathbf{I} \\ \sum_{n/(2e) < b \leq n/e} \frac{\theta_2(b)}{be} & \text{if } e \text{ is even and } \mathbf{T} = \mathbf{QI} \\ \sum_{\substack{n/(2e) < b \leq n/e \\ b \text{ even}}} \frac{\theta_2(b)}{be} & \text{if } e \text{ is odd and } \mathbf{T} = \mathbf{QI}. \end{cases} \quad (3.9)$$

First consider the case where  $\mathbf{T} = \mathbf{QI}$  with  $e$  even. Bounds on  $\theta_2$  are obtained in a similar fashion to the bounds on  $\theta$  in (3.3). An upper bound  $\theta_2(b) \leq 1 - 1/p^{t+j}$  is obtained by excluding the elements in  $\mathbb{F}_{q^{be}}^*$  with order not divisible by  $p$ , which comprise the unique subgroup of index  $(q^{be} - 1)_p = p^{t+j}$ , where  $p^j = (b)_p$  and  $p^t = (q^e - 1)$ . A lower bound for  $\theta_2(b)$  is obtained by considering the proportion of elements  $\zeta \in \mathbb{F}_{q^{be}}^*$  that are not Galois conjugate to  $\zeta^{-1}$  and have  $be/2$  Galois conjugates over  $\mathbb{F}_{q^2}$ , and then subtracting the proportion of elements in  $\mathbb{F}_{q^{be}}^*$  with order not divisible by  $p$ . A lower bound for the former proportion is given by Lemma 4.3(ii) with  $q$  replaced by  $q^2$  and  $\ell = be/2$ ,

yielding  $\theta_2(b) > 1 - 1/p^{t+j} - 3/q^{be/2} > 1 - 1/p^{t+j} - 3/q^{n/4}$ . In other words, (3.3) holds if  $\theta$  is replaced by  $\theta_2$ . Moreover, the corresponding (second) sum in (3.9) is the same as the sum in (3.2), except with  $\theta$  replaced by  $\theta_2$ . It follows that we can obtain the same bounds for  $|Q(p; \mathbf{QI}; G)|/|G|$  as we obtained for  $|Q(p; \mathbf{I}; \text{GL}_n(q))|/|\text{GL}_n(q)|$ . That is, the calculations in (3.4)–(3.5) and (3.6)–(3.8), with  $\mathbf{I}$  replaced by  $\mathbf{QI}$  and  $\theta$  replaced by  $\theta_2$  in (3.6), yield (1.1) with the constants  $c, \kappa, \alpha, \beta$  given in lines 5 and 6 of Table 2, which are identical to lines 1 and 2, respectively.

Now let  $\mathbf{T} = \mathbf{QI}$  with  $e$  odd. The only difference from the previous case is that now the corresponding sum in (3.9) is restricted to even values of  $b$ . For  $e > n/p$  this means that we proceed as in (3.4)–(3.5) but with  $\mathbf{I}$  replaced by  $\mathbf{QI}$  and the proportion  $P(n/e, 1)$  replaced by  $P'(n/e, 1)$ , where bounds on  $P'$  are given in Lemma 4.4(ii). The result is that  $\ln(2)$  is replaced by  $\ln(2)/2$ , and thus in line 7 of Table 2 as compared with line 5, the value of  $c$  is divided by 2. For  $e \leq n/p$  we use (3.6)–(3.8) with  $\mathbf{I}$  replaced by  $\mathbf{QI}$ ,  $\theta$  replaced by  $\theta_2$  in (3.6),  $P(n/e, p^j)$  replaced by  $P'(n/e, p^j)$  for  $j = 0, \dots, i$ , and the sums over  $b$  restricted to even values of  $b$ . So now when applying Lemma 4.6 we set  $f_j = P'(n/e, p^j)$  and hence  $k_1 = \ln(2)/2$  instead of  $k_1 = \ln(2)$ . The result is that the values of both  $c$  and  $\beta$  are divided by 2 in line 8 of Table 2 as compared with line 6.

It remains to consider the case where  $\mathbf{T} = \mathbf{I}$ , which arises (only) when  $e \equiv 2 \pmod{4}$ . The basic steps are similar to those in the preceding cases, with a few differences in the details. An upper bound for  $\theta_1(b)$  is obtained by excluding the elements of  $\mathbb{Z}_{q^{be/2}+1}$  with order not divisible by  $p$ , which comprise the unique subgroup of  $\mathbb{Z}_{q^{be/2}+1}$  of index  $(q^{be/2}+1)_p$ . By Lemma 4.1(iv),  $(q^{be/2}+1)_p = p^{t+j}$ , where  $p^j = (b)_p$  and  $p^t = (q^e - 1)$ , and so  $\theta_1(b) \leq 1 - 1/p^{t+j}$ . A lower bound for  $\theta_1(b)$  is obtained by considering the proportion of elements  $\mathbb{Z}_{q^{be/2}+1}^* < \mathbb{F}_{q^{be}}^*$  that have  $be/2$  Galois conjugates over  $\mathbb{F}_{q^2}$ , and then subtracting the proportion of elements with order not divisible by  $p$ . A lower bound on former proportion is given by Lemma 4.3(iv) with  $\ell = be/2$ , yielding  $\theta_1(b) > 1 - 1/p^{t+j} - 3/q^{be/4} > 1 - 1/p^{t+j} - 3/q^{n/4}$ , where the second inequality holds since  $be = 2m > n$  in the present case. In summary, we have

$$1 - \frac{1}{p^{t+j}} - \frac{3}{q^{n/4}} < \theta_1(b) \leq 1 - \frac{1}{p^{t+j}}. \quad (3.10)$$

We now estimate the corresponding (first) sum in (3.9). First suppose that  $e > 2n/p$ . Then  $b \leq 2n/e < p$  for all  $b$  in the corresponding sum in (3.9), and so  $j = 0$  for all  $b$  in the bounds for  $\theta_1$  in (3.10). Hence, in the notation of Lemma 4.4(iii),

$$\left(1 - \frac{1}{p^t} - \frac{3}{q^{n/4}}\right) \frac{2P''(2n/e, 1)}{e} < \frac{|Q(p; \mathbf{I}; G)|}{|G|} \leq \left(1 - \frac{1}{p^t}\right) \frac{2P''(2n/e, 1)}{e}.$$

Applying Lemma 4.4(iii) and a calculation similar to (3.4)–(3.5) yields (1.1) with  $c, \kappa, \alpha, \beta$  as in line 3 of Table 2.

Now suppose that  $e \leq 2n/p$ . Let  $i$  be the positive integer such that  $p^i \leq n/e < p^{i+1}$ . Each  $b$  in the first sum in (3.9) satisfies  $(b)_p = p^j$  for some  $j \in$

$\{0, \dots, i\}$ , and so the bounds in (3.10) depend on  $j = j(b)$ . We write

$$\frac{|Q(p; \mathbf{I}; G)|}{|G|} = \sum_{j=0}^i \left( \sum_{\substack{n/e < b \leq 2n/e \\ (b)_p = p^j, b \text{ odd}}} \frac{2\theta_1(b)}{be} \right)$$

and apply (3.10) to obtain the following inequalities analogous to (3.7)–(3.8):

$$\begin{aligned} \frac{|Q(p; \mathbf{I}; G)|}{|G|} &\leq \frac{2}{e} \sum_{j=0}^{i-1} \left( 1 - \frac{1}{p^{t+j}} \right) (P''(2n/e, p^j) - P''(2n/e, p^{j+1})) \\ &\quad + \frac{2}{e} \left( 1 - \frac{1}{p^{t+i}} \right) P''(2n/e, p^i), \\ \frac{|Q(p; \mathbf{I}; G)|}{|G|} &> \frac{2}{e} \sum_{j=0}^{i-1} \left( 1 - \frac{1}{p^{t+j}} - \frac{3}{q^{n/4}} \right) (P''(2n/e, p^j) - P''(2n/e, p^{j+1})) \\ &\quad + \frac{2}{e} \left( 1 - \frac{1}{p^{t+i}} - \frac{3}{q^{n/4}} \right) P''(2n/e, p^i). \end{aligned}$$

Applying Lemma 4.4(iii), and Lemma 4.6 with  $f_j = P''(2n/e, p^j)$  and hence  $\ell = 2n/e$ ,  $k_1 = \ln(2)/2$ ,  $k_2 = 2$  (and  $p, q, t, i$  as already defined), we obtain (1.1) with constants as in line 4 of Table 2.

### 3.3 $G = \mathbf{Sp}_{2n}(q)$ , $\mathbf{SO}_{2n+1}(q)$ or $\mathbf{SO}_{2n}^\pm(q)$

First suppose that  $G = \mathbf{Sp}_{2n}(q)$  or  $G = \mathbf{SO}_{2n+1}(q)$ , and let  $d = 2n$  or  $d = 2n+1$ , respectively. From the discussion in Section 2.2, here  $C$  is a quokka class for  $Q(p; \mathbf{T}; G)$  if and only if  $T_C^F$  has a direct factor  $A_\lambda \cong Z_{q^\lambda \pm 1}$  and  $p$  divides  $q^\lambda \pm 1$  for some  $\lambda$  such that  $m = 2\lambda$  satisfies  $d/2 < m \leq d$ , which for the integer  $\lambda$  is equivalent to  $n/2 < \lambda \leq n$ . The image  $\sigma(g) \in S_n$  of an element  $g$  of  $C$  must have a cycle of length  $\lambda$  as described below.

For a negative  $\lambda$ -cycle,  $p$  must divide  $q^\lambda + 1$ . According to Lemma 4.1(ii), this occurs if and only if  $e$  divides  $2\lambda$  and  $e$  does not divide  $\lambda$ . So  $e$  must be even, and we need  $\lambda = be/2$  for some odd  $b$  with  $n/2 < \lambda \leq n$ , namely  $n/e < b \leq 2n/e$ . By [12, Lemma 4.2(a)], the proportion of elements in  $W$  with a negative cycle of length  $\lambda$  is half the proportion of elements in  $S_n$  with a cycle of length  $\lambda$ , and hence, by (3.1), is equal to  $1/(2\lambda)$ . The corresponding  $(p, 2\lambda)$ -abundant elements in  $T_C^F$  are of irreducible type, since a negative cycle of length  $\lambda$  in  $S_n$  corresponds to a single cycle of length  $2\lambda$  in  $W = S_2 \wr S_n$ . An element of  $T_C^F$  is  $(p, 2\lambda)$ -abundant irreducible if and only if its  $A_\lambda$ -component has an eigenvalue in  $\mathbb{F}_{q^{2\lambda}}$  with  $2\lambda$  Galois conjugates over  $\mathbb{F}_q$  and multiplicative order divisible by  $p$ . Denoting by  $\theta^-(b)$  the proportion of elements in  $\mathbb{Z}_{q^{be/2+1}} < \mathbb{F}_{q^{be}}^*$  with multiplicative order divisible by  $p$  and  $be$  Galois conjugates over  $\mathbb{F}_q$ , we obtain (3.12) below for the cases  $G = \mathbf{Sp}_{2n}(q)$  and  $G = \mathbf{SO}_{2n+1}$ .

For a positive  $\lambda$ -cycle we need  $p$  to divide  $q^\lambda - 1$ , that is,  $e$  must divide  $\lambda$  (by Lemma 4.1(i)). So  $\lambda = be$  for some  $b$  with  $n/2 < \lambda \leq n$ , namely  $n/(2e) < b \leq n/e$ . By [12, Lemma 4.2(a)] and (3.1), the proportion of elements in  $W$  with a positive cycle of length  $\lambda$  is  $1/(2\lambda)$ . The corresponding  $(p, 2\lambda)$ -abundant are of quasi-irreducible type, since a positive cycle of length  $\lambda$  in  $S_n$  corresponds to two cycles of length  $\lambda$  in  $W = S_2 \wr S_n$ . An element of  $T_C^F$  is  $(p, 2\lambda)$ -abundant quasi-irreducible if and only if its  $A_\lambda$ -component has an eigenvalue  $\zeta \in \mathbb{F}_{q^\lambda}$  with multiplicative order divisible by  $p$  such that  $\zeta$  and  $\zeta^{-1}$  are not Galois conjugate and have together  $2\lambda$  Galois conjugates over  $\mathbb{F}_q$ . Hence, denoting by  $\theta^+(b)$  the proportion of elements  $\zeta \in \mathbb{F}_{q^{be}}^*$  that have multiplicative order divisible by  $p$ , are not Galois conjugate to  $\zeta^{-1}$  and have  $be$  Galois conjugates over  $\mathbb{F}_q$ , we obtain (3.11) below for the cases  $G = \mathrm{Sp}_{2n}(q)$  and  $G = \mathrm{SO}_{2n+1}$ .

Now consider  $G = \mathrm{SO}_{2n}^\pm(q)$ , and recall the discussion at the end of Section 2.2. A slight modification to the above argument is required. An  $F$ -conjugacy class  $C$  in  $W$  is a quokka class for  $Q(p; \mathbf{T}; G)$  if and only if, for an element  $g$  of  $C$ , or of  $Cw_n$  for  $G = \mathrm{SO}_{2n}^-(q)$ , the image  $\sigma(g) \in S_n$  satisfies the same conditions as for  $G = \mathrm{Sp}_{2n}(q)$  and in addition the total number of negative cycles is even if  $G = \mathrm{SO}_{2n}^+(q)$  or odd if  $G = \mathrm{SO}_{2n}^-(q)$ . In particular, if  $G = \mathrm{SO}_{2n}^+(q)$  then we cannot have a single negative cycle of length  $n$ , and for  $G = \mathrm{SO}_{2n}^-(q)$  we cannot have a single positive cycle of length  $n$ . The result is that when  $\lambda = n$ , the Weyl group proportion  $1/(2\lambda) = 1/(2n)$  used above changes to  $1/n$  if  $\mathbf{T} = \mathbf{I}$ ,  $G = \mathrm{SO}_{2n}^-(q)$  or  $\mathbf{T} = \mathbf{QI}$ ,  $G = \mathrm{SO}_{2n}^+(q)$  and to 0 in the other two cases. This is reflected in equations (3.11) and (3.12) below.

Let us summarise. Let  $G = \mathrm{Sp}_{2n}(q)$ ,  $\mathrm{SO}_{2n+1}(q)$  or  $\mathrm{SO}_{2n}^\pm(q)$  for the remainder of this section. For the  $p$ -abundant quasi-irreducible elements in  $G$ , which arise for all values of  $e$ , we have

$$\frac{|Q(p; \mathbf{QI}; G)|}{|G|} = \left( \sum_{n/(2e) < b \leq n/e} \frac{\theta^+(b)}{2be} \right) + \frac{\lambda^+ \theta^+(n/e)}{2n}, \quad (3.11)$$

where

$$\lambda^+ = \begin{cases} 1 & \text{if } G = \mathrm{SO}_{2n}^+(q) \text{ and } e \text{ divides } n \\ -1 & \text{if } G = \mathrm{SO}_{2n}^-(q) \text{ and } e \text{ divides } n \\ 0 & \text{otherwise.} \end{cases}$$

For even values of  $e$  (only), we also have  $p$ -abundant irreducible elements arising, and the proportion of these elements is given by

$$\frac{|Q(p; \mathbf{I}; G)|}{|G|} = \left( \sum_{\substack{n/e < b \leq 2n/e \\ b \text{ odd}}} \frac{\theta^-(b)}{be} \right) + \frac{\lambda^- \theta^-(2n/e)}{2n}, \quad (3.12)$$

with

$$\lambda^- = \begin{cases} -1 & \text{if } G = \mathrm{SO}_{2n}^+(q) \text{ and } 2n/e \text{ is an odd integer} \\ 1 & \text{if } G = \mathrm{SO}_{2n}^-(q) \text{ and } 2n/e \text{ is an odd integer} \\ 0 & \text{otherwise.} \end{cases}$$

It remains to estimate the above sums in order to complete the proof of Theorem 1.2. First consider (3.11). Begin by observing that

$$\left| \frac{|Q(p; \mathbf{QI}; G)|}{|G|} - \sum_{n/(2e) < b \leq n/e} \frac{\theta^+(b)}{2be} \right| \leq \frac{1}{2n}.$$

The sum in the above inequality is the same as the sum in (3.2), except for the factor of  $1/2$  and the fact that  $\theta$  has been replaced by  $\theta^+$ . Moreover, the inequalities in (3.3) also hold if  $\theta$  is replaced by  $\theta^+$ , as can be seen by applying part (ii) Lemma 4.3 with  $\ell = be$  instead of part (i). It follows that  $|Q(p; \mathbf{QI}; G)|/|G|$  satisfies the same bounds as those obtained in Section 3.1 for  $|Q(p; \mathbf{I}; \mathrm{GL}_n(q))|/|\mathrm{GL}_n(q)|$ , except that we must divide by 2 and then add  $1/(2n)$  to the upper bound and subtract  $1/(2n)$  from the lower bound. Thus, comparing lines 11 and 12 of Table 2 with lines 1 and 2, the values of  $c$  and  $\beta$  are halved, while halving  $\alpha = 1$  and then adding  $1/2$  yields  $\alpha = 1$  again.

Now consider (3.12), noting that

$$\left| \frac{|Q(p; \mathbf{I}; G)|}{|G|} - \sum_{\substack{n/e < b \leq 2n/e \\ b \text{ odd}}} \frac{\theta^-(b)}{be} \right| \leq \frac{1}{2n}.$$

The sum above is the same as the first sum in (3.9), except divided by 2 and with  $\theta_1$  replaced by  $\theta^-$ . Moreover, the bounds for  $\theta_1$  in (3.10) also hold with  $\theta_1$  replaced by  $\theta^-$ , which is verified by applying part (iii) of Lemma 4.3 with  $\ell = be/2$  instead of part (iv). It follows that  $|Q(p; \mathbf{I}; G)|/|G|$  satisfies the same bounds as those obtained in Section 3.2 for  $|Q(p; \mathbf{I}; \mathrm{GU}_n(q))|/|\mathrm{GU}_n(q)|$  in the case  $e \equiv 2 \pmod{4}$ , except that we must divide by 2 and then add  $1/(2n)$  to the upper bound and subtract  $1/(2n)$  from the lower bound. In other words, comparing lines 9 and 10 of Table 2 with lines 3 and 4, the values of  $c$  and  $\beta$  are halved, while halving  $\alpha = 2$  and then adding  $1/2$  yields  $\alpha = 3/2$ .

## 4 Technical results

Here we collect the various technical results used in Section 3. Section 4.1 gives the results used to estimate the proportions  $\theta$ ,  $\theta_1$ ,  $\theta_2$  and  $\theta^\pm$  in equations (3.2), (3.9), (3.11) and (3.12). Section 4.2 collects the estimates of the various sums used to complete the proof of Theorem 1.2.

### 4.1 Torus proportions

**Lemma 4.1** *Let  $q$  be a prime power,  $p$  a prime not dividing  $q$ ,  $e$  the smallest positive integer such that  $p$  divides  $q^e - 1$ , and write  $p^t = (q^e - 1)_p$ . Then*

- (i)  *$p$  divides  $q^m - 1$  if and only if  $e$  divides  $m$ ,*

- (ii)  $p$  divides  $q^m + 1$  if and only if  $e$  divides  $2m$  and  $e$  does not divide  $m$ ,
- (iii) if  $p \neq 2$  and  $b$  is a positive integer with  $(b)_p = p^j$  then  $(q^{be} - 1)_p = p^{t+j}$ ,
- (iv) if  $p \neq 2$ ,  $e$  is even and  $b$  is an odd positive integer with  $(b)_p = p^j$  then  $(q^{be/2} + 1)_p = p^{t+j}$ .

**Proof.** For proofs of (i) and (ii), see [12, Lemma 4.5].

(iii) If  $p$  does not divide  $b$ , namely  $j = 0$ , then  $(q^{be} - 1)_p = p^t$  since

$$\frac{q^{be} - 1}{q^e - 1} = 1 + q^e + \dots + q^{e(b-1)} \equiv b \pmod{p} \not\equiv 0 \pmod{p}.$$

The proof is completed by induction on  $j$ . Claim 1 below says that without loss of generality we may assume that  $b = p^j$ . (Here we also note that  $p$  does not divide  $e$ , since  $e$  divides  $p - 1$  and is therefore coprime to  $p$ .) It then suffices to apply Claim 2 below with  $k = q^{ep^j}$  and  $r = t + j$ .

*Claim 1.* Let  $v, v'$  be positive integers such that  $v'$  is a multiple of  $v$  and  $(v')_p = (v)_p$ . Then  $(q^{v'} - 1)_p = (q^v - 1)_p$ .

*Proof of Claim 1.* Write  $p^r = (q^v - 1)_p$ . If  $v' = v\ell$  with  $\ell$  not divisible by  $p$  then  $(q^{v'} - 1)/(q^v - 1) = 1 + q^v + \dots + q^{(\ell-1)v} \equiv \ell \pmod{p^r}$ , and therefore  $(q^{v'} - 1)/(q^v - 1) \not\equiv 0 \pmod{p}$ .

*Claim 2.* If  $k$  is a positive integer with  $(k - 1)_p = p^r$ , where  $r \geq 1$ , then  $(k^p - 1)_p = p^{r+1}$ .

*Proof of Claim 2.* Write  $k' = 1 + k + \dots + k^{p-1}$ . Then  $k^p - 1 = (k - 1)k'$  and hence  $(k^p - 1)_p = p^r(k')_p$ . So it suffices to check that  $(k')_p = p$ . Since  $k = 1 + p^r y$  for some  $y$  that is not divisible by  $p$ , we have

$$\begin{aligned} k' &= 1 + (1 + yp^r) + (1 + yp^r)^2 + \dots + (1 + yp^r)^{p-1} \\ &= p + yp^r(1 + 2 + \dots + (p-1)) + zp^{2r} \end{aligned}$$

for some  $z$ . So

$$k' = p + y \frac{p-1}{2} p^{r+1} + zp^{2r},$$

and since  $r \geq 1$  it follows that  $k'$  is divisible by  $p$  but not by  $p^2$ , namely  $(k')_p = p$ .

(iv) Since  $b$  is odd,  $e$  does not divide  $be/2$ . So (i) implies that  $p$  does not divide  $q^{be/2} - 1$ , namely that  $(q^{be/2} - 1)_p = 1$ , and then (iii) yields  $(q^{be/2} + 1)_p = (q^{be} - 1)_p / (q^{be/2} - 1)_p = p^{t+j}$ .  $\square$

**Lemma 4.2** *Let  $q$  be a prime power and  $\ell$  a positive integer with  $\ell \geq 2$ . If  $\zeta \in \mathbb{F}_{q^\ell}$  is Galois conjugate to  $\zeta^{-1}$  over  $\mathbb{F}_q$  then either  $\zeta$  lies in a proper subfield of  $\mathbb{F}_{q^\ell}$  or  $\ell$  is even and  $\zeta$  lies in cyclic the subgroup of  $\mathbb{F}_{q^\ell}^*$  of order  $q^{\ell/2} + 1$ .*

**Proof.** Let  $i$  denote the least nonnegative integer such that  $\zeta^{q^i} = \zeta^{-1}$ . If  $i = 0$  then  $\zeta^2 = 1$  and hence  $\zeta = \pm 1$  lies in every proper subfield of  $\mathbb{F}_{q^\ell}$ . Suppose now that  $i > 0$ . Then  $\zeta^{q^{2i}} = \zeta$ , or  $\zeta^{q^{2i}-1} = 1$ . Since  $\zeta^{q^\ell-1} = 1$ , this implies that



$\zeta^{q^{\gcd(\ell, 2i)} - 1} = \zeta^{\gcd(q^\ell - 1, q^{2i} - 1)} = 1$ . Write  $k = \gcd(\ell, 2i)$ . If  $k \neq \ell$  then  $k < \ell$  and  $k$  divides  $\ell$ , so  $\zeta$  lies in  $\mathbb{F}_{q^k} < \mathbb{F}_{q^\ell}$  and the result holds. This leaves  $k = \ell$ , in which case  $i = r\ell/2$  for some positive integer  $r$ . Observe that  $\zeta^{q^{j\ell}} = \zeta$  for any nonnegative integer  $j$ . If  $r$  is even then  $\zeta^{-1} = \zeta^{q^i} = \zeta^{q^{\ell(r/2)}} = \zeta$  and hence  $i = 0$ , which we already considered above. Suppose therefore that  $r$  is odd, in which case  $\ell$  is even. Then  $i = (r-1)/2 \cdot \ell + \ell/2$  and  $\zeta^{-1} = \zeta^{q^i} = \zeta^{q^{(r-1)/2 \cdot \ell + \ell/2}} = \zeta^{q^{\ell/2}}$ . By the minimality of  $i$ , it follows that  $r = 1$  and so  $i = \ell/2$ . Thus  $\zeta$  satisfies  $\zeta^{q^{\ell/2} + 1} = 1$  and therefore lies in the cyclic subgroup of  $\mathbb{F}_{q^\ell}^*$  of order  $q^{\ell/2} + 1$ .  $\square$

**Lemma 4.3** *Let  $\ell$  be a positive integer. Then the following hold:*

- (i) *The proportion of elements in  $\mathbb{F}_{q^\ell}^*$  with  $\ell$  Galois conjugates over  $\mathbb{F}_q$  is greater than  $1 - 3/q^{\ell/2}$ .*
- (ii) *The proportion of elements  $\zeta \in \mathbb{F}_{q^\ell}^*$  that are not Galois conjugate to  $\zeta^{-1}$  and have  $\ell$  Galois conjugates over  $\mathbb{F}_q$  is greater than  $1 - 3/q^{\ell/2}$ .*
- (iii) *The proportion of elements in  $\mathbb{Z}_{q^\ell+1} < \mathbb{F}_{q^{2\ell}}^*$  with  $2\ell$  Galois conjugates over  $\mathbb{F}_q$  is greater than  $1 - 3/q^{\ell/2}$ .*
- (iv) *For  $\ell$  odd, the proportion of elements in  $\mathbb{Z}_{q^\ell+1} < \mathbb{F}_{q^{2\ell}}^*$  with  $\ell$  Galois conjugates over  $\mathbb{F}_{q^2}$  is greater than  $1 - 3/q^{\ell/2}$ .*

**Proof.** (i) Write  $A = \mathbb{F}_{q^\ell}^*$ . The elements in  $A$  with  $\ell$  Galois conjugates over  $\mathbb{F}_q$  are precisely those that lie in no proper subfield of  $A$ . Hence, denoting by  $\rho(A)$  the set of all elements of  $A$  that lie in some field  $K$  with  $\mathbb{F}_q \leq K < \mathbb{F}_{q^\ell}^*$ , we must show that  $|A \setminus \rho(A)|/|A| > 1 - 3/q^{\ell/2}$ . If  $\ell = 1$  then this inequality holds vacuously because  $\rho(A)$  is empty, so we now assume that  $\ell \geq 2$ . If  $\zeta$  is an element of some field  $K$  with  $\mathbb{F}_q \leq K < \mathbb{F}_{q^\ell}^*$  then there is a prime divisor  $r$  of  $\ell$  such that  $\zeta \in K \leq \mathbb{F}_{q^{\ell/r}}^*$ . Hence

$$|\rho(A)| \leq \sum_{\substack{r \text{ prime} \\ r|\ell}} (q^{\ell/r} - 1) \leq \left( \sum_{j=1}^{\lfloor \ell/2 \rfloor} q^j \right) - 1 < \frac{q^{\ell/2+1} - 1}{q - 1} - 1 \leq 2q^{\ell/2} - 2.$$

So  $|\rho(A)|/|A| < 2/q^{\ell/2}$ , and thus  $|A \setminus \rho(A)|/|A| > 1 - 2/q^{\ell/2}$ . In particular,  $|A \setminus \rho(A)|/|A| > 1 - 3/q^{\ell/2}$  as claimed.

(ii) Write  $A = \mathbb{F}_{q^\ell}^*$  again. If  $\ell$  is odd then by Lemma 4.2 the proportion of elements of  $A$  that we are seeking is precisely as in (i), and is thus greater than  $1 - 3/q^{\ell/2}$ . Now suppose that  $\ell$  is even. Let  $\rho(A)$  denote the set of elements in  $A$  that lie in either a proper subfield of  $A$  or in the cyclic subgroup of  $A$  of order  $q^{\ell/2} + 1$ . By Lemma 4.2 the proportion that we are seeking is  $|A \setminus \rho(A)|/|A|$ . From the proof of (i) we know that at most  $2q^{\ell/2} - 2$  elements lie in a proper subfield of  $A$ . So  $|\rho(A)| \leq (2q^{\ell/2} - 2) + q^{\ell/2} = 3q^{\ell/2} - 2$ , and thus  $|A \setminus \rho(A)|/|A| > 1 - 3/q^{\ell/2}$ .

(iii) Write  $A = \mathbb{Z}_{q^\ell+1}$  and let  $\rho(A)$  denote the set of all elements of  $A$  that lie in some field  $K$  with  $\mathbb{F}_q \leq K < \mathbb{F}_{q^{2\ell}}^*$ . The proportion that we are seeking is  $|A \setminus \rho(A)|/|A|$ . For  $\zeta \in \rho(A)$  we have  $\zeta \in \mathbb{F}_{q^{2\ell/r}}^*$  for some prime  $r$  dividing  $2\ell$ , and so  $|\zeta|$  divides

$$\gcd(q^\ell + 1, q^{2\ell/r} - 1) = \begin{cases} q^{\ell/r} + 1 & \text{if } r \text{ is odd} \\ \gcd(2, q - 1) & \text{if } r = 2. \end{cases}$$

If  $\ell = 1$  or  $\ell$  is a power of 2 then  $r = 2$  (only) and hence  $|\rho(A)|/|A| \leq 2/(q^\ell + 1)$ . Then  $|A \setminus \rho(A)|/|A| \geq 1 - 2/q^\ell > 1 - 2/q^{\ell/2}$  and the result holds. If  $\ell$  is not a power of 2 then

$$\begin{aligned} |\rho(A)| &\leq 1 + \frac{2}{q^\ell + 1} + \sum_{\substack{r \text{ prime} \\ r|\ell, r \geq 3}} q^{\ell/r} \leq \frac{2}{q^\ell + 1} + \sum_{j=0}^{\lfloor \ell/3 \rfloor} q^j \\ &\leq \frac{2}{q^\ell + 1} + \frac{q^{\ell/3+1} - 1}{q - 1} \leq \frac{2}{q^\ell + 1} + 2q^{\ell/3} < 3q^{\ell/3}. \end{aligned}$$

So  $|A \setminus \rho(A)|/|A| > 1 - 3q^{\ell/3}/q^\ell = 1 - 3/q^{2\ell/3} > 1 - 3/q^{\ell/2}$ .

(iv) Write  $A = \mathbb{Z}_{q^\ell+1}$  and let  $\rho(A)$  denote the set of all elements of  $A$  that lie in some field  $K$  with  $\mathbb{F}_{q^2} \leq K < \mathbb{F}_{q^{2\ell}}^*$ . The result holds if  $\ell = 1$  since then  $\rho(A)$  is empty, so we may assume that  $\ell \geq 3$  (with  $\ell$  odd). In this case an element in  $\mathbb{F}_{q^{2\ell}}$  with  $\ell$  Galois conjugates over  $\mathbb{F}_{q^2}$  has  $2\ell$  Galois conjugates over  $\mathbb{F}_q$ , and so the result holds by (iii).  $\square$

## 4.2 Sums

**Lemma 4.4** *Let  $\ell$  be a real number and  $r$  an integer with  $1 \leq r \leq \ell$ . Write*

$$P(\ell, r) = \sum_{\substack{\ell/2 < b \leq \ell \\ r|b}} \frac{1}{b}, \quad P'(\ell, r) = \sum_{\substack{\ell/2 < b \leq \ell \\ r|b, b \text{ even}}} \frac{1}{b}, \quad P''(\ell, r) = \sum_{\substack{\ell/2 < b \leq \ell \\ r|b, b \text{ odd}}} \frac{1}{b},$$

*assuming further that  $r$  is odd in the definition of  $P''(\ell, r)$ . Then*

- (i)  $|P(\ell, r) - \ln(2)/r| \leq 1/\ell$ ,
- (ii)  $|P'(\ell, r) - \ln(2)/(2r)| \leq 1/\ell$ ,
- (iii)  $|P''(\ell, r) - \ln(2)/(2r)| \leq 2/\ell$ .

**Proof.** We make use of the following easily verified inequalities, in which  $k_1, k_2$  are positive integers with  $2 \leq k_1 \leq k_2$ , and  $x$  is a real number with  $x > -1$ :

$$\ln\left(\frac{k_2+1}{k_1}\right) \leq \sum_{j=k_1}^{k_2} \frac{1}{j} \leq \ln\left(\frac{k_2}{k_1-1}\right), \quad \frac{x}{x+1} \leq \ln(1+x) \leq x. \quad (4.1)$$

First consider the case where  $r = 1$ . The above inequalities yield

$$\begin{cases} -1/(\ell+1) & \text{if } \ell \text{ is even} \\ 0 & \text{if } \ell \text{ is odd} \end{cases} \leq P(\ell, 1) - \ln(2) \leq \begin{cases} 0 & \text{if } \ell \text{ is even} \\ 1/(\ell+1) & \text{if } \ell \text{ is odd,} \end{cases}$$

where in the case of the upper bound for odd  $\ell \geq 3$ , we extract the  $b = (\ell+1)/2$  term from the sum before applying the relevant inequality. It follows that  $|P(\ell, 1) - \ln(2)| \leq 1/(\ell+1) < 1/\ell$ . If  $r = 2$  then writing  $b = jr$  in the definition of  $P(\ell, r)$  gives

$$P(\ell, r) = \frac{1}{r} \sum_{\ell/(2r) < j \leq \ell/r} \frac{1}{j} = \frac{P(\lfloor \ell/r \rfloor, 1)}{r},$$

and it follows that  $|P(\ell, r) - \ln(2)/r| \leq 1/(r(\lfloor \ell/r \rfloor + 1)) \leq 1/\ell$ . This completes the proof of (i). Assertions (ii) and (iii) follow: for  $r = 1$  we note that  $P'(\ell, 1) = P(\lfloor \ell/2 \rfloor, 1)/2$  and  $P''(\ell, 1) = P(\ell, 1) - P'(\ell, 1)$ , and then for  $r = 2$  we write  $P'(\ell, r) = P'(\lfloor \ell/r \rfloor, 1)/r$  and  $P''(\ell, r) = P''(\lfloor \ell/r \rfloor, 1)/r$ .  $\square$

**Lemma 4.5** *Let  $p, q, t$  be real numbers with  $p \geq 3, q \geq 2, t \geq 1$ . Then  $1 - 1/p^t - 3/q^{n/4} > 0$  for all integers  $n$  with  $n \geq 9$ .*

**Proof.** Since  $p^t \geq 3$  and  $q \geq 2$ , the required inequality holds provided that  $2/3 - 3/q^{n/4} > 0$ , namely  $n > 4 \log(9/2)/\log(2) \approx 8.68$ .  $\square$

**Lemma 4.6** *Let  $p$  be a real number with  $p \geq 3$  and  $i, \ell$  positive integers with  $p^i \leq \ell < p^{i+1}$ . For  $j \in \{0, \dots, i\}$ , suppose that  $f_j$  are real numbers such that*

$$\frac{k_1}{p^j} - \frac{k_2}{\ell} \leq f_j \leq \frac{k_1}{p^j} + \frac{k_2}{\ell}$$

*for some positive real numbers  $k_1, k_2$ . Then*

(i) *for any real number  $t$  with  $t \geq 1$ ,*

$$\left(1 - \frac{1}{p^{t+i}}\right) f_i + \sum_{j=0}^{i-1} \left(1 - \frac{1}{p^{t+j}}\right) (f_j - f_{j+1}) < \left(1 - \frac{1}{p^{t-1}(p+1)}\right) k_1 + \frac{k_2}{\ell};$$

(ii) *if  $n$  is an integer with  $n \geq 9$  then for any real  $q, t$  with  $q \geq 2$  and  $t \geq 1$ ,*

$$\begin{aligned} & \left(1 - \frac{1}{p^{t+i}} - \frac{3}{q^{n/4}}\right) f_i + \sum_{j=0}^{i-1} \left(1 - \frac{1}{p^{t+j}} - \frac{3}{q^{n/4}}\right) (f_j - f_{j+1}) \\ & > \left(1 - \frac{1}{p^{t-1}(p+1)}\right) k_1 - \left(k_2 + \frac{k_1}{p^t}\right) \frac{1}{\ell} - \frac{3k_1}{q^{n/4}}. \end{aligned}$$

**Proof.** (i) We have

$$\begin{aligned}
& \left(1 - \frac{1}{p^{t+i}}\right) f_i + \sum_{j=0}^{i-1} \left(1 - \frac{1}{p^{t+j}}\right) (f_j - f_{j+1}) = \left(1 - \frac{1}{p^t}\right) f_0 + \frac{p-1}{p^t} \sum_{j=1}^i \frac{f_j}{p^j} \\
& \leq \left(1 - \frac{1}{p^t}\right) \left(k_1 + \frac{k_2}{\ell}\right) + \frac{p-1}{p^t} \sum_{j=1}^i \frac{1}{p^j} \left(\frac{k_1}{p^j} + \frac{k_2}{\ell}\right) \\
& = \left(1 - \frac{1}{p^t} + \frac{p-1}{p^t} \sum_{j=1}^i \frac{1}{p^{2j}}\right) k_1 + \left(1 - \frac{1}{p^t} + \frac{p-1}{p^t} \sum_{j=1}^i \frac{1}{p^j}\right) \frac{k_2}{\ell} \\
& = \left(1 - \frac{1}{p^{t-1}(p+1)} - \frac{1}{p^t(p+1)p^{2i}}\right) k_1 + \left(1 - \frac{1}{p^t p^i}\right) \frac{k_2}{\ell} \\
& < \left(1 - \frac{1}{p^{t-1}(p+1)}\right) k_1 + \frac{k_2}{\ell}.
\end{aligned}$$

(ii) Set  $E(n) = 3/q^{n/4}$  for brevity. By Lemma 4.5, the assumption  $n \geq 9$  implies that  $1 - 1/p^t - E(n) > 0$ . This validates the first inequality below. The second inequality below follows from the assumption  $p^{i+1} > \ell$ , and the third from  $p^i \leq \ell$ , which implies that  $\ell \geq p$  (since  $i \geq 1$ ):

$$\begin{aligned}
& \left(1 - \frac{1}{p^{t+i}} - E(n)\right) f_i + \sum_{j=0}^{i-1} \left(1 - \frac{1}{p^{t+j}} - E(n)\right) (f_j - f_{j+1}) \\
& = \left(1 - \frac{1}{p^t} - E(n)\right) f_0 + \frac{p-1}{p^t} \sum_{j=1}^i \frac{f_j}{p^j} \\
& > \left(1 - \frac{1}{p^t} - E(n)\right) \left(k_1 - \frac{k_2}{\ell}\right) + \frac{p-1}{p^t} \sum_{j=1}^i \frac{1}{p^j} \left(\frac{k_1}{p^j} - \frac{k_2}{\ell}\right) \\
& = \left(1 - \frac{1}{p^t} + \frac{p-1}{p^t} \sum_{j=1}^i \frac{1}{p^{2j}} - E(n)\right) k_1 - \left(1 - \frac{1}{p^t} + \frac{p-1}{p^t} \sum_{j=1}^i \frac{1}{p^j} - E(n)\right) \frac{k_2}{\ell} \\
& = \left(1 - \frac{1}{p^{t-1}(p+1)} - \frac{1}{p^t(p+1)p^{2i}} - E(n)\right) k_1 - \left(1 - \frac{1}{p^t p^i} - E(n)\right) \frac{k_2}{\ell} \\
& > \left(1 - \frac{1}{p^{t-1}(p+1)} - \frac{1}{p^{t-2}(p+1)\ell^2} - E(n)\right) k_1 - \frac{k_2}{\ell} \\
& = \left(1 - \frac{1}{p^{t-1}(p+1)}\right) k_1 - \frac{k_2}{\ell} - \frac{k_1}{p^{t-2}(p+1)\ell^2} - k_1 E(n) \\
& > \left(1 - \frac{1}{p^{t-1}(p+1)}\right) k_1 - \left(k_2 + \frac{k_1}{p^t}\right) \frac{1}{\ell} - k_1 E(n).
\end{aligned}$$

□

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